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CONTINUUM DESCRIPTION OF HYSTERESIS DAMPING OF VIBRATIONS

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Abstract—A continuum description of hysteresis damping of vibrations is constructed based on the assumption that the hysteresis in the elastic response is due to the occurrence of small plastic strains below the over-all elastic limit of the material. The associated constitutive law describing elasticity with hysteresis is developed assuming the plastic strain rate for loading is dependent only on the stress and stress rate. It is shown that the constitutive law so developed yields the well-known Kimball–Lovell quadratic damping law for sufficiently low stress levels and that it provides additional terms for describing damping when higher stress levels are involved.

The theory is applied to the free torsional vibration of wires and is shown to yield exponential amplitude decay for low stress levels. At higher stresses, the decay is found to be greater than this limiting exponential form in accordance with long standing experience. Detailed theoretical results are compared with measurements of the torsional response of a soft copper wire and the description is found to be excellent.

INTRODUCTION

It has long been recognized that the two major causes of vibration damping in metals are rate-insensitive hysteresis in the elastic response of the material [1–10] and rate-sensitive heat flow in the material itself [10, 11]. In the special case where only shear deformations are involved, thermal damping will be absent owing to the absence of volume changes during the motion, and we are then left with only rate-insensitive hysteresis damping. This is of course the situation with torsional vibrations, and, to the extent that microscopic thermal effects are negligible, the damping can thus be expected to be independent of the frequency of the motion and dependent only on its amplitude. Even with other vibrations where thermal effects can arise, it generally happens that the hysteresis damping dominates, in which case these same remarks will also apply [3, 5–7, 9].

The discovery of rate-insensitive hysteresis damping may properly by attributed to Lord Kelvin [1, 2] who first reported its existence in 1865 and expanded upon it in 1878. Using what we now call a Kelvin–Voigt viscoelastic model, he pointed out that, if the damping forces were proportional to the rate of change of shape of the material, as they are for viscous fluids, the fractional decay in amplitude per period of motion must vary inversely with the period. Experiments with free torsional vibrations of a hanging wire having a rigid mass attached to its free end showed, however, no such dependence when the period of the motion was reduced from 3.51 to 2.47 sec. In fact, only a slight dependence on the period was noted at all, and Kelvin concluded that the damping was nothing like that of a viscous fluid but rather like that expected from imperfections (hysteresis) in the elasticity of the wire.

Some 50 years later, Hopkinson and Trevor-Williams [3] considered the problem afresh and showed using rough estimates of the hysteresis in the static response that their measurements of the energy dissipated during forced longitudinal vibration of a steel bar at a frequency of 136 Hz could be fully accounted for by the hysteresis. The following year, Hopkinson's student Rowett [4] extended this work to the free torsional vibration of a hollow steel tube having a massive disc attached at one end and oscillating at a frequency of 67 Hz. In addition to measuring the decay in amplitude of the vibrations, he also made detailed measurements of the static hysteresis and demonstrated in an admirable way that the loss in energy per cycle of vibration could be closely predicted by the static hysteresis measurements.

This was essentially the situation when, in 1926, Kimball and Lovell [5] reported damping measurements from a series of experiments with horizontal rotating beams having fixed vertical loading at one end. They found for frequencies ranging from about 8 to 30 Hz, that the accompanying small lateral deflections resulting from damping effects were insensitive to the frequency, and they accordingly concluded, as in the earlier work, that the damping force was sensibly independent of the frequency. On the basis of their work, they also proposed for a simple damping law that the energy loss per unit volume of material in a cycle of vibration be taken proportional to the square of the stress amplitude. They emphasized, however, that such a law was intended to apply only for relatively low stress amplitudes and that, when the stress amplitude is a sizable percent of the yield stress of the material, a more complex law must apply.

Since the appearance of the work of Kimball and Lovell, the rate insensitive character of hysteresis damping has been confirmed in a number of investigations and for a variety of materials. It seems now to be generally accepted that the quadratic law of Kimball and Lovell is valid for sufficiently low stress levels but fails altogether when, as they originally warned, the stress amplitude approaches the yield stress of the material. Attempts have been made to generalize this law to a simple power law expression in order to account for higher stress-level data but nothing definite can be said of this approach since one power-law representing the energy loss at one stress level will not do so at another. This, of course, is quite different from the quadratic law of Kimball and Lovell which may be regarded as the limiting form of a more general relation. The situation with the quadratic law is, in fact, analogous to that for Hooke's law of perfect elasticity which is the limiting small-strain form of a more general non-linear relation (see, for example, my recent book *Theory and Practice of Solid Mechanics* [12]).

In order to make any further significant gains in describing hysteresis damping, it is clear that a continuum description of some generality is required so that damping measurements for all stress amplitudes of interest can be incorporated within a single framework. This suggests therefore that our attention should be directed not at the energy loss itself but rather at the underlying mechanics.

The explanation now generally accepted for hysteresis damping in metals is based on the documented occurrence of small plastic strains at stresses below the ordinary yield stress of the material, thus causing the elastic response on loading to differ slightly from that on unloading. This difference and the attendant loss of energy is altogether negligible for a single cycle of vibration and becomes important only by accumulation over many cycles. Up to the present, however, no general continuum law connecting these strains with the stresses has yet been proposed. If this description were available, we should expect it to yield by calculation the Kimball–Lovell quadratic law of energy loss for small stress amplitudes and to provide a means for incorporating data for higher stress amplitudes directly into the general description. In the present paper, we provide such a continuum description for one-dimensional stress states and illustrate its applicability by consideration of experimental data from free torsional vibrations.

CONSTITUTIVE LAW FOR ELASTIC RESPONSE WITH HYSTERESIS

The purpose of a constitutive law of deformation is to connect the stresses and strains within a given material. When the total strain is made up of elastic and plastic parts, as in the case of elasticity with hysteresis, we cannot, however, expect the stress and strain to be uniquely related, since the strains on loading will not be the same as those on unloading. Thus, to describe elasticity with plastic-strain hysteresis, we must deal not with the strains themselves but rather with increments of the strain or, equivalently, with the time rate of change of the strain. This, of course, is what we also have to do in classical plasticity theory.

Restricting attention to one-dimensional stress states, which includes, of course, the important cases of longitudinal, torsional and flexural vibrations, we may therefore assume that the total time rate of change of the strain $\dot{\epsilon}$ is determined at any instant by the sum of the respective elastic and plastic strain rates. Now, the elastic strain rate is determined by differentiation of Hooke's law and is given directly as $\dot{\sigma}/C$ where $\dot{\sigma}$ denotes the time rate of change of the stress σ and C denotes the appropriate elastic modulus for normal or shear stresses. On the basis of our understanding of plastic deformation, we may further assume that the plastic strain rate on loading is determined by some function of the stress and stress rate. But the relation between the plastic strain rate and the stress and stress rate cannot be fully general, since rate-independence requires that the plastic strain rate vary directly with the stress rate. Otherwise, a doubling of the stress rate would not result in a doubling of the strain rate, and rate effects would be represented. Thus, with this condition and the previous elastic strain-rate relation, our general constitutive law describing elasticity with hysteresis takes the form

$$\dot{\epsilon} = \frac{\dot{\sigma}}{C}\left[1 + f\left(\frac{\sigma}{Y}\right)\right] \tag{1}$$

where f denotes some general function and Y denotes the appropriate ordinary yield stress of the material for normal or shear stresses, introduced here for convenience so as to make the stress dimensionless.

In addition to this equation, we may also assume, on the basis of our general understanding of plasticity, that plastic deformation occurs only when positive work is being done on the material; that is, only during loading. With this condition, the function f in eqn (1) is then required to satisfy

$$\begin{aligned} f &= f, \text{ when } \sigma\dot{\epsilon} > 0 \\ f &= 0, \text{ when } \sigma\dot{\epsilon} \leq 0. \end{aligned} \tag{2}$$

Finally, we assume that the hysteresis is the same for both positive and negative cycles of loading so that the function f must be even in the stress. Expanding eqn (1) into a Taylor series about zero stress we therefore obtain

$$\dot{\epsilon} = \frac{\dot{\sigma}}{C}\left[1 + \beta_1 + \beta_2\left(\frac{\sigma}{Y}\right)^2 + \beta_3\left(\frac{\sigma}{Y}\right)^4 + \cdots\right] \tag{3}$$

where β_1, β_2, etc. are coefficients which, in view of eqn (2) must satisfy

$$\begin{aligned} &\beta_1 = \beta_1, \quad \beta_2 = \beta_2, \text{ etc. when } \sigma\dot{\epsilon} > 0 \\ &\beta_1 = 0, \quad \beta_2 = 0, \text{ etc. when } \sigma\dot{\epsilon} \leq 0. \end{aligned} \tag{4}$$

This simple relation is derived from two basic assumptions only: that the total strain rate is made up of elastic and plastic parts and that the plastic strain rate for loading is dependent only on the stress and rate of stress. Let us now examine an immediate consequence of the relation, namely, the expression for the energy loss per unit volume of material that occurs during a single cycle of vibration.

Consider first the case when the stress is increased from zero to some positive stress level σ_0. During this loading, the strain–stress relation is determined from eqn (3) by integration as

$$\epsilon - \epsilon^* = \frac{\sigma}{C}\left[1 + \beta_1 + \frac{1}{3}\beta_2\left(\frac{\sigma}{Y}\right)^2 + \frac{1}{5}\beta_3\left(\frac{\sigma}{Y}\right)^4 + \cdots\right] \tag{5}$$

where ϵ^* denotes the strain when $\sigma = 0$. The work w_1 expended per unit volume of material is thus

$$w_1 = \int_{\epsilon^*}^{\epsilon_0} \sigma \,\mathrm{d}\epsilon = \int_0^{\sigma_0} (\epsilon_0 - \epsilon)\,\mathrm{d}\sigma \tag{6}$$

where ϵ_0 denotes the strain associated with σ_0. Using eqn (5) to obtain the difference $\epsilon_0 - \epsilon$, we find, on substituting the result into this last equation, that

$$w_1 + \frac{1}{2}(1 + \beta_1)\frac{\sigma_0^2}{C} + \frac{1}{4}\beta_2\frac{\sigma_0^4}{CY^2} + \frac{1}{6}\beta_3\frac{\sigma_0^6}{CY^4} + \cdots. \tag{7}$$

Now consider complete unloading from the stress state σ_0. The work w_2 recovered in this case per unit volume of material is expressible as

$$w_2 = \frac{1}{2}\frac{\sigma_0^2}{C} \tag{8}$$

so that the net work per unit volume done per half cycle is $w_1 - w_2$ and that done per cycle is $w = 2(w_1 - w_2)$; or, in dimensionless form,

$$\frac{W}{Y} = \frac{Y}{C}\left[\beta_1\left(\frac{\sigma_0}{Y}\right)^2 + \frac{1}{2}\beta_2\left(\frac{\sigma_0}{Y}\right)^4 + \frac{1}{3}\beta_3\left(\frac{\sigma_0}{Y}\right)^6 + \cdots\right]. \tag{9}$$

From this result, we see immediately that for sufficiently small stress levels all terms in the brackets except the first may be neglected, and the energy loss per unit volume of material in a cycle of loading will then be proportional simply to the square of the stress amplitude σ_0. This, of course, is the Kimball–Lovell quadratic damping relation proposed over 50 years ago on purely empirical grounds. We also see that when the stresses are not small enough to make the quadratic law applicable, higher-order terms are available to allow description by the constitutive law of eqn (3).

We may call the coefficients $\beta_1, \beta_2, \ldots$ the first, second, etc. coefficients of hysteresis and we note that the Kimball–Lovell coefficient relating energy loss to the square of the stress, when multiplied by the associated elastic modulus gives directly the first coefficient β_1.

APPLICATION TO FREE TORSIONAL VIBRATION OF WIRES

With a view toward comparison with some experimental results, we now apply our theory to the case of free torsional vibration of wires. Here the stress will not be uniform over the cross section of the wire and we must make further calculations in order to describe its response.

Taking σ and ϵ in this case to represent shearing stress and shearing strain, we have the net moment acting on any section given by

$$M = 2\pi \int_0^{r_0} \sigma r^2 \, \mathrm{d}r \tag{10}$$

where r denotes radial distance and r_0 denotes the radius of the wire.

For an explicit expression for the stress σ, we may solve eqn (5) to obtain

$$\frac{\sigma}{C} = \alpha_1(\epsilon - \epsilon^*) - \alpha_2(\epsilon - \epsilon^*)^3 + \cdots \tag{11}$$

where

$$\alpha_1 = \frac{1}{1+\beta_1}, \quad \alpha_2 = \frac{1}{3}\frac{\beta_2 C^2}{Y^2(1+\beta_1)^4}.$$

Remembering from solid mechanics that the shear strains ϵ and ϵ^* and the associated angles of twist ϕ and ϕ^* are related through the radial position r and the distance x along the wire by the kinematic equations

$$\epsilon = r\frac{\partial \phi}{\partial x}, \quad \epsilon^* = r\frac{\partial \phi^*}{\partial x} \tag{12}$$

we find from eqns (10) and (11) that

$$M = b_1\alpha_1\frac{\partial \bar{\phi}}{\partial x} - b_2\alpha_2\left(\frac{\partial \bar{\phi}}{\partial x}\right)^3 + \cdots \tag{13}$$

where $\bar{\phi} = \phi - \phi^*$ and b_1 and b_2 are defined by

$$b_1 = \frac{\pi C r_0^4}{2}, \quad b_2 = \frac{\pi C r_0^6}{3}.$$

If we now solve eqn (13) for $\partial\bar{\phi}/\partial x$ and integrate over the total length L of the wire, we find

$$\phi - \phi^* = \frac{1+\beta_1}{K_1}M + \frac{\beta_2}{3K_2}M^3 + \cdots \tag{14}$$

where

$$K_1 = \frac{b_1}{L}, \quad K_2 = \frac{b_1{}^4 Y^2}{b_2 L C^2}.$$

Here, ϕ represents the angle of twist at one end of the wire, the other end being assumed fixed, and ϕ^* is the value of ϕ when $M = 0$. Finally, taking the derivative of this expression with respect to time, we obtain an equation analogous to eqn (3) in the form

$$\dot{\phi} = \frac{\dot{M}}{K_1}\left[1 + \beta_1 + \frac{K_1\beta_2}{K_2} M^2 + \cdots\right] \tag{15}$$

where, as before, the coefficients β_1, β_2, etc. must vanish during unloading.

The expression for the work done per cycle of vibration W may be determined in a manner similar to that used to obtain eqn (9). We find

$$W = \frac{\beta_1}{K_1} M_0{}^2 + \frac{1}{2}\frac{\beta_2}{K_2} M_0{}^4 + \cdots \tag{16}$$

where M_0 denotes the amplitude of the moment.

The decay in amplitude of the motion may now be determined from this last relation and the energy statement

$$K_1 A \frac{\mathrm{d}A}{\mathrm{d}n} = -W \tag{17}$$

where A denotes the amplitude of the oscillations after n cycles of motion. Combining these last two equations and putting

$$M_0 = K_1 A \tag{18}$$

we thus find, on retaining only the first two terms in eqn (16), that

$$A = A_0[1 + \lambda - \lambda\, \mathrm{e}^{-2\beta_1 n}]^{-1/2}\, \mathrm{e}^{-\beta_1 n} \tag{19}$$

where A_0 denotes the initial amplitude and λ is given by

$$\lambda = \frac{1}{2}\frac{\beta_2}{\beta_1}\frac{K_1{}^3}{K_2} A_0{}^2. \tag{20}$$

From this equation, we see that if β_2 is set equal to zero so that only the first term in eqn (16) is retained, the predicted amplitude decay will be exponential, the decay being governed by the first coefficient of hysteresis β_1. Thus, β_1 also has the interpretation of being the logarithmic decrement of classical vibration theory.

When, however, the next higher-order term in eqn (16) is retained, we then see that the decay will only become exponential after a sufficiently large number of vibrations have occured to make the exponential term within the brackets negligibly small. During the earlier part of the vibrations, our amplitude equation indicates, in fact, that for positive β_2 the decay will be greater than this limiting decay. If, therefore, we have vibration-damping data of sufficiently large amplitude to make the second term in eqn (16) important, and we interpret the data in terms of only first-order theory, we would thus conclude *erroneously* that the first coefficient β_1 (or logarithmic decrement) must vary with the amplitude, its value being greater during the early part of vibrations than during the later stage. Remarkably enough, this is precisely what has been done over the past many years, the effect now generally being referred to as the amplitude effect[7, 13].

COMPARISON WITH EXPERIMENTS ON SOFT COPPER WIRE

We may examine the details of the above theoretical results by considering measurements of the free torsional vibrations of a soft copper wire having one end fixed and the other attached to a rigid cylinder of known mass, as in the case of Kelvin's original experiments. The wire used in the present work had a length of 73 cm (28.8 in) and a diameter of 0.081 cm (0.032 in). Tensile tests on similar wires revealed a tensile yield stress of 42.9 MN/m^2 (6220 psi) so that using the Maxwell–Mises yield condition, the yield stress in shear can be taken approximately as 24.8 MN/m^2 (3590 psi). The elastic modulus C of the wire was estimated from the period of the torsional oscillations as 37.7 KN/mm^2 (5.46×10^6 psi).

Amplitude-decay data were obtained by releasing the rigid mass from an initial angle of twist and measuring the resulting angular motion. This was done with the help of a calibrated photo-electric device that allowed continuous monitoring of the amount of light passing through a curved V-slit attached to the base of the rigid cylinder. The experiments were carried out in air, but the effect of the air resistance on the observed damping was entirely negligible for the periods of motion considered. Rough calculations using viscous fluid theory indicated, in fact, that the energy loss resulting from the air drag was only about 0.10% of that actually observed so that any attempt to correct for the air resistance would be meaningless.

Figure 1 shows amplitude-decay data so measured for anitial angle of twist A_0 of 32° and for end masses of amount 1 kg (weight 2.2 lb) and 0.26 kg (weight 0.57 lb). Both of these cylindrical masses were made of brass with diameters of 1.91 cm (0.75 in). The larger mass had a height of 10.2 cm (4 in) and the smaller had a height one-fourth this, i.e. 2.54 cm (1 in). The periods of the motion for the 1 kg and 0.26 kg masses were measured as 1.80 and 0.93 sec, respectively. The same wire was used in all experiments. Each data point shown in Fig. 1 represents the average from two tests. It will be seen that, in agreement with earlier investigations, there is no significant dependence on the period of the motion so that the damping may be regarded as rate insensitive.

Also shown in Fig. 1 are predictions from our theoretical amplitude relation of eqn (19) when values of β_1 and λ (which depends on β_2) are taken as

$$\beta_1 = 3.63 \times 10^{-3}, \quad \lambda = 2.32.$$

The agreement is seen to be excellent.

To illustrate the non-exponential decay of the amplitude in these experiments, values of $\log(A/A_0)$ have been plotted vs the number of cycles of vibration in Fig. 2. It will be seen that the data fail altogether to fall on a straight line from the origin of the coordinates so that the measurements are not at all represented by first-order exponential decay. On the other hand, however, when compared with the predictions of eqn (19) using the above values of β_1 and λ, the agreement is then seen to be remarkably good. We note that the maximum stress in these experiments is almost 50% of the yield stress in shear so that the Kimball–Lovell quadratic law and associated exponential decay should not be expected to apply.

For an independent examination, we may consider measurements made with the same wire and under the same conditions as above except that the initial angle of twist is reduced from 32

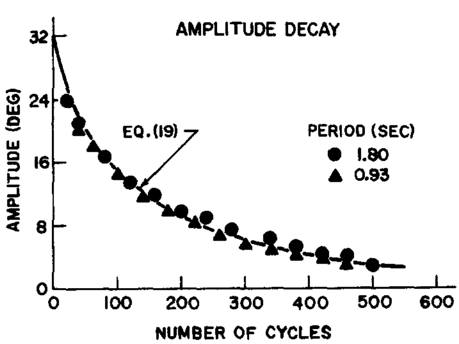

Fig. 1.

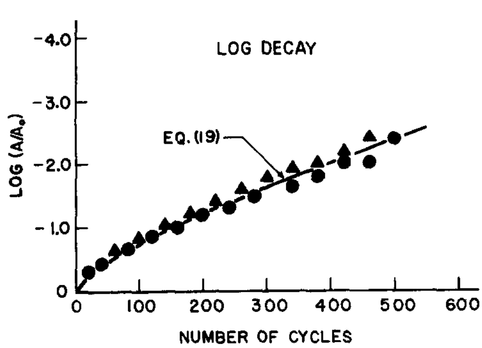

Fig. 2.

Fig. 1. Amplitude-decay data from free torsional oscillations of soft copper wire having initial amplitude of 32°.

Fig. 2. Logarithmic plot illustrating non-exponential decay of the data of Fig. 1.

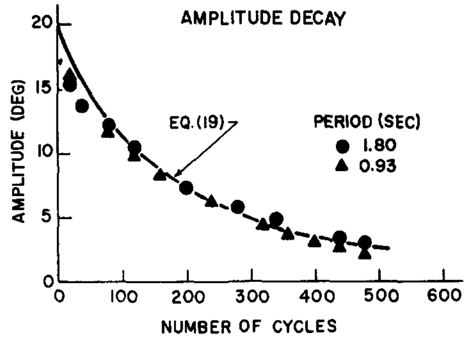

Fig. 3. Amplitude-decay data from free torsional oscillations of soft copper wire having an initial amplitude of 20°. The theoretical curve is determined using parameters established from the data of Fig. 1.

to 20°. These measurements are shown in Fig. 3 along with predictions from eqn (19) based on the above values of β_1 and λ, the value of λ being adjusted for the different initial amplitude in accordance with eqn (20); that is,

$$\lambda = 2.32\,(20/32)^2 = 0.906.$$

The agreement, as before, is seen to be remarkably good. It is to be emphasized that the value of λ for this case is not chosen arbitrarily, but rather is *predicted* from eqn (20) and the previous value of λ. Thus, the agreement seen in Fig. 3 is indicative of the validity of the theoretical squared-amplitude dependence of eqn (20).

We note finally that the value of the second coefficient of hysteresis β_2 may be determined directly from eqn (20) and the above value of λ. In terms of the elastic limit Y and modulus C, we find

$$\beta_2 = (2.62 \times 10^5)(Y/C)^2.$$

If we neglect the axial stress in the wire resulting from the weight of the end cylinder, the elastic limit in shear will, as previously noted, be 24.8 MN/m^2 and β_2 will equal 0.113. Alternatively, if we account for the axial stress using the Maxwell–Mises yield condition, we find for the 1 kg end mass that the yield stress is reduced to 22.3 MN/m^2 so that β_2 will equal 0.091. Similarly, for the 0.26 kg end mass, the yield stress will be 24.6 MN/m^2 and β_2 will equal 0.112. In either case, the value of β_2/Y^2 will, of course, be the same, this being all that enters into our theoretical equations.

SUMMARY

It has been our purpose here to construct a continuum description of hysteresis damping of vibrations from fundamental considerations of the constitutive law governing elasticity with hysteresis. We have seen that the governing rate-independent constitutive law for one-dimensional stress states can be established from the assumption that the rate of strain of the material consists of an ordinary elastic part, dependent on the stress rate, and a plastic part dependent on both the stress and stress rate. When expanded into a Taylor series about zero stress, this law has been found to yield the well known Kimball–Lovell quadratic energy-loss relation for low stress levels and to provide additional terms for describing higher stress situations. This result is of considerable significance considering the fundamental basis on which it is established and should not be confused with earlier work where *a priori* assumptions are introduced concerning the shape of the hysteresis loop in a cycle of loading and where unknown parameters are assigned by experiment (see Lazan[13]).

We note also that the constitutive relation developed here is itself of considerable interest in solid mechanics since it gives for the first time a description of plasticity below the conventional yield stress of the material. The condition for plastic strains in this case is not related to a critical yield stress value, but rather to the requirement that positive work be done on the material.

The constitutive law has been applied to the specific problem of the free torsional vibration of wires and the amplitude decay has been worked out for the case where only the first two terms in the Taylor expansion are retained. The amplitude decay is predicted to be exponential after a sufficiently large number of vibrations (i.e. for sufficiently low stress levels) but not necessarily so during the earlier, higher-stress stage. This effect has long been recognized experimentally. Of major importance here, however, is the fact that, for the first time, a theoretical basis for the effect is given. Significantly, it is found from the present theory by going but one step farther than that required to obtain the Kimball–Lovell quadratic energy relation and associated exponential decay.

The theoretical amplitude relation has been examined experimentally using measurements from the free torsional vibration of a soft copper wire. The agreement has been found to be excellent for both the exponential and non-exponential parts of the amplitude decay. The non-linear theoretical dependence of the decay on the initial amplitude has also been examined by contrasting data associated with an initial angle of twist of 32° with data associated with an initial 20° twist. The agreement between theory and experiment has again been found to be excellent.

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